## On an Alternative Approach to the Relation between Bosons and Fermions: Employing Clifford Space

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## Abstract

We further explore the idea that physics takes place in Clifford space which should be considered as a generalization of spacetime. Following the old observation that spinors can be represented as members of left ideals of Clifford algebra, we point out that the transformations which mix bosons and fermions could be represented by means of operators acting on Clifford algebra-valued (polyvector) fields. A generic polyvector field can be expanded either in terms of bosonic or in terms of fermionic fields. In particular, a scalar field can transform into a mixture of bosonic and/or fermionic fields.

The idea that Clifford algebra might provide a clue to the unification of fundamental interactions has been explored by a number of researchers [1, 2, 3], [4]–[6]. Crawford considered local transformations which involve all  $2^n$  generators of Clifford algebra acting on column spinors. The compensating gauge fields behaved as Yang-Mills fields. Instead of column spinors one can represent spinors geometrically, as members of left or right ideals of Clifford algebra [7, 11]. Chisholm used geometric spinors, but he took only spinor of one left ideal, and left out those belonging to other left ideals. The first who took advantage of the full Clifford algebra in describing wave functions was Pezzaglia. He proposed that wave functions be Clifford algebra valued objects, called *polyvectors*, satisfying the generalized Klein-Gordon or the Dirac equation. In his particular model the wave function depended

on vector and bivector coordinates. In ref. [8] the polyvector Dirac equation in which the polyvector wave function depended on polyvector coordinates, wheareas in ref. [9] the analogous wave function in the context of the Klein-Gordon equation was considered. Those ideas were subsequently further developed in refs. [4, 5, 10], and it was realized that a polyvector field can contain either bosons or fermions, or both at once, and thus provided a realization of a kind of supersymmetry.

In this letter I will demonstrate how the transformations, proposed in by Chisholm and Crawford [1], when operating on polyvector fields ("wave functions"), can change bosons into fermions, and vice versa.

Let X be a coordinate polyvector

$$X = x^{M} \gamma_{M} = \sum_{r=1}^{n} x^{\mu_{1} \dots \mu_{n}} \gamma_{\mu_{1} \dots \mu_{r}} , \quad \mu_{1} < \mu_{2} < \dots < \mu_{r}$$
 (1)

expanded in terms of the basis  $\{\gamma_M\} = (\mathbf{1}, \gamma_{\mu_1}, \gamma_{\mu_1 \mu_2}, ..., \gamma_{\mu_1 ... \mu_r})$  of the Clifford algebra generated by  $\gamma_{\mu}$ ,  $\mu = 1, 2, ..., n$  that satisfy

$$\gamma_{\mu} \cdot \gamma_{\nu} \equiv \frac{1}{2} (\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu}) = g_{\mu\nu} \mathbf{1}$$
 (2)

The basis  $\{\gamma_M\}$  spans a manifold, called *Clifford space*, shortly *C*-space. The metric of *C* space is defined as the scalar product

$$G_{MN} = \gamma_M^{\ddagger} * \gamma_N \tag{3}$$

Here '‡' denotes the reversion, that is the operation which reverses the order of the generators  $\gamma_{\mu}$  (for example,  $\gamma^{\ddagger}_{\mu_1\mu_2\mu_3} = \gamma_{\mu_3\mu_2\mu_1}$ ), whilst '\*' denotes the scalar product between two Clifford numbers A and B

$$A * B = \langle AB \rangle_0 \tag{4}$$

In general, C-space is curved and we distinguish between the coordinate basis elements  $\gamma_M$  and local basis elements  $\gamma_A$  [12]. For simplicity, in this paper we confine our consideration to flat C-space, therefore  $\gamma_M = \gamma_A$ .

Let us consider a field theory in C-space, by introducing a polyvector field

$$\Phi = \phi^M(X)\gamma_M = \phi^A(X)\gamma_A \tag{5}$$

We assume that the components  $\phi^A$  are in general complex valued. We interprete the imaginary unit i in the way that is usual in quantum theory, namely that i lies outside the

Clifford algebra of spacetime and hence commutes with all  $\gamma_M$ . This is different from the point of view hold by many researchers of the geometric calculus based on Clifford algebra (see, e.g., [13, 14]). They insist that i has to be defined geometrically, so it must be one of the elements of the set  $\{\gamma_A\}$ , such that its square equals -1. An alternative interpretation, also often assumed, is that i is the pseudoscalar unit of a higher dimensional space. For instance, if our spacetime is assumed to be 4-dimensional, then i is the pseudoscalar unit of a 5-dimensional space. The problem then arises about a physical interpretation of the extra dimension. This is not the case that we adopt. Instead we adopt the view, first proposed in [5], that i is the bivector of the 2-dimensional phase space  $P_2$ , spanned by  $e_q$ ,  $e_p$ , so that  $Q \in P_2$  is equal to  $Q = e_q e_q + p e_p$ ,  $e_q Q = q + i p$ ,  $i = e_q e_p$ . So our i is also defined geometrically, but the space we employ differs form the spaces usually considered in defining i. Taking into account that there are four spacetime dimensions, the total phase space is thus the direct product  $M_4 \times P_2 = P_8$ , so that any element  $Q \in P_8$  is equal to  $Q = x^{\mu}e_{\mu}e_{p} + p^{\mu}e_{\mu}e_{p}$ ,  $e_{q}Q = (x^{\mu} + ip^{\mu})e_{\mu}$ . This can then be generalized to Clifford space by replacing  $x^{\mu}$ ,  $p^{\mu}$  by the corresponding Clifford space variables  $x^{M}$ ,  $p^{M}$ . In a classical theory, we can just consider  $x^{\mu}$  only (or  $x^{M}$  only), and forget about  $p^{\mu}$  ( $p^{M}$ ), since  $x^{\mu}$  and  $p_{\mu}$  are independent. In quantum theory,  $x^{\mu}$  and  $p_{\mu}$  ( $x^{M}$  and  $p_{M}$ ) are complementary variables, therefore we cannot formulate a theory without at least implicitly involving the presence of momenta  $p_{\mu}$  ( $p_{M}$ ). Consequently, wave functions are in generally complex valued. Hence the occurrence of i in quantum mechanics is not perplexing, it arises from phase space. We adopt here the conventional interpretation of quantum mechanics; no hidden variables, Böhmian potential, etc., just the Born statistical interpretation and Bohr-Von Neumann projection postulate.

Following Teitler [11] (whose work built on Riesz [7]) we introduce spinors as follows. Instead of the basis  $\{\gamma_A\}$  one can consider another basis, which is obtained after multiplying  $\gamma_A$  by 4 independent primitive idempotents [11]

$$P_i = \frac{1}{4} (\mathbf{1} + a_i \gamma_A + b_i \gamma_B + c_i \gamma_C) , \quad i = 1, 2, 3, 4$$
 (6)

such that

$$P_i = \frac{1}{4} (\mathbf{1} + a_i \gamma_A) (\mathbf{1} + b_i \gamma_B) , \quad \gamma_A \gamma_B = \gamma_C , \quad c_i = a_i b_i$$
 (7)

Here  $a_i$ ,  $b_i$ ,  $c_i$  are complex numbers chosen so that  $P_i^2 = P_i$ . For explicit and systematic construction see [11, 15].

By means of  $P_i$  we can form minimal ideals of Clifford algebra. A basis of a left (right) minimal ideal is obtained by taking one of  $P_i$  and multiply it from the left (right) with all 16 elements  $\gamma_A$  of the algebra:

$$\gamma_A P_i \in \mathcal{I}_i^L$$
,  $P_i \gamma_A \in \mathcal{I}_i^R$  (8)

Here  $\mathcal{I}_i^L$  and  $\mathcal{I}_i^R$ , i = 1, 2, 3, 4 are four independent minimal left and right ideals, respectively. For a fixed i there are 16 elements  $P_i\gamma_A$ , but only 4 amongst them are different, the remaining elements are just repetition of those 4 different elements.

Let us denote those different elements  $\xi_{\alpha i}$ ,  $\alpha = 1, 2, 3, 4$ . They form a basis of the *i*-th left ideal. Every Clifford number can be expanded either in terms of  $\gamma_A = (1, \gamma_{a_1}, \gamma_{a_1 a_2}, \gamma_{a_1 a_2 a_3}, \gamma_{a_1 a_2 a_3 a_4})$  or in terms of  $\xi_{\alpha i} = (\xi_{\alpha 1}, \xi_{\alpha 2}, \xi_{\alpha 3}, \xi_{\alpha 4})$ :

$$\Phi = \phi^A \gamma_A = \Psi = \psi^{\alpha i} \xi_{\alpha i} = \psi^{\tilde{A}} \xi_{\tilde{A}} \tag{9}$$

In the last step we introduced a single spinor index  $\tilde{A}$  which runs over all 16 basis elements that span 4 independent left minimal ideals. Explicitly, eq. (9) reads

$$\Psi = \psi^{\tilde{A}} \xi_{\tilde{A}} = \psi^{\alpha 1} \xi_{\alpha 1} + \psi^{\alpha 2} \xi_{\alpha 2} + \psi^{\alpha 3} \xi_{\alpha 3} + \psi^{\alpha 4} \xi_{\alpha 4}$$
 (10)

Eq.(9) or (10) represents a direct sum of four independent 4-component spinors, each living in a different left ideal  $\mathcal{I}_i^L$ . The polyvector field  $\Phi$  can be expanded either in term of r-vector basis elements  $\gamma_A$  (bosonic fields) or in terms of the spinor basis elements  $\xi_{\tilde{A}}$  (fermionic fields).

A generic transformation in C-space which maps a polyvector  $\Psi$  into another polyvector  $\Psi'$  is given by

$$\Psi' = R\Psi S \tag{11}$$

where

$$R = e^{\gamma_A \alpha^A}$$
 and  $S = e^{\gamma_A \beta^A}$  (12)

Here  $\alpha^A$  and  $\beta^A$  are parameters of the transformation. Requiring that the transformations (11) should leave the quadratic form  $\Psi^{\ddagger} * \Psi$  invariant. So we have  $\psi'^{\ddagger} * \Psi' = \langle \psi'^{\ddagger} \Psi' \rangle_S = \langle S^{\ddagger} \Psi'^{\ddagger} R^{\ddagger} R \Psi S \rangle_S = \langle \Psi^{\ddagger} \Psi \rangle_S = \Psi^{\ddagger} * \Psi$ , provided that  $R^{\ddagger} R = 1$  and  $S^{\ddagger} S = 1$ . Explicitly, the quadratic form reads  $\Psi^{\ddagger} * \Psi = \psi^{*\tilde{A}} \psi^{\tilde{B}} z_{\tilde{A}\tilde{B}}$ , where  $z_{\tilde{A}\tilde{B}} = \xi_{\tilde{A}}^{\ddagger} * \xi_{\tilde{B}}$  is the spinor metric.

The transformations that are usually considered are those for which  $\beta^A = -\alpha^A$ , i.e.,  $R = R^{-1}$ , but here we allow for more general transformations (11). They mix bosonic

and fermionic field. This can be seen on the following example. Let  $\phi \mathbf{1}$  be a scalar valued field. Operating on it by a transformation R from the left we obtain a new field which is a mixture of fields of different grades r:

$$\Phi' = R\phi \mathbf{1} = \phi e^{\gamma_A \alpha^A} \mathbf{1} = \phi e^{\xi_{\tilde{A}} \alpha^{\tilde{A}}} \mathbf{1} = \phi'^A \gamma_A = \psi'^{\tilde{A}} \xi_{\tilde{A}}$$
(13)

From a scalar we thus obtain a polyvector. But a polyvector can written, according to eq.(9), as a mixture of fermionic fields. Therefore, our transformation R acting on  $\phi$  has a role of a supersymetric transformations. For other r-vector fields, i.e., the fields with definite grade r, we have:

$$R\phi^a \gamma_a = \phi^a C_a{}^B \gamma_B = \phi'' B \gamma_B = \psi''^{\tilde{B}} \xi_{\tilde{B}}$$
 (14)

$$R\phi^{a_1 a_2} \gamma_{a_1 a_2} = \phi^{a_1 a_2} C_{a_1 a_2}{}^B \gamma_B = \phi''' B \gamma_B = \psi''' \tilde{B} \xi_{\tilde{B}}$$
 (15)

$$\vdots (16)$$

The above equations say that an r-vector field, r = 0, 1, 2, 3, 4, transforms into a superposition of r-vector fields, which in turn is a superposition of spinor fields belonging to different left ideals.

In examples (13)–(15) a bosonic field is transformed into a mixture of fermionic fields. The inverse transformations are also possible and they read:

$$R\psi^{\alpha 1}\xi_{\alpha 1} = \psi^{\alpha 1}K_{\alpha 1}{}^{\tilde{B}}\xi_{\tilde{B}} = \psi'^{\tilde{B}}\xi_{\tilde{B}} = \phi'^{B}\gamma_{B}$$

$$\tag{17}$$

$$R\psi^{\alpha 2}\xi_{\alpha 2} = \psi^{\alpha 2}K_{\alpha 2}{}^{\tilde{B}}\xi_{\tilde{B}} = \psi''^{\tilde{B}}\xi_{\tilde{B}} = \phi''^{B}\gamma_{B}$$

$$\tag{18}$$

$$\vdots (19)$$

In eqs. (13)–(18) we have particular cases of a general transformation (11) which transforms a polyvector field  $\Phi$  into another polyvector field  $\Phi'$ , or equivalently, a generalized spinor field  $\Psi$  into another generalized spinor field  $\Psi'$ :

Such view on spinors and supersymmetries has potentially profound implications for further development of field theory, and in particular, of string theory. The procedure that we described above, can be applied to generalized point particles and strings as well.

We can envisage that physical objects are described in terms of  $x^M = (\sigma, x^{\mu}, x^{\mu\nu}, ...)$ . The first straightforward possibility is to introduce a single parameter  $\tau$  and consider a mapping

$$\tau \to x^M = X^M(\tau) \tag{20}$$

where  $X^M(\tau)$  are 16 embedding functions that describe a worldline in C-space. From the point of view of C-space,  $X^M(\tau)$  describe a wordline of a "point particle": at every value of  $\tau$  we have a *point* in C-space. But from the perspective of the underlying 4-dimensional spacetime,  $X^M(\tau)$  describe an extended object, sampled by the center of mass coordinates  $X^{\mu}(\tau)$  and the coordinates  $X^{\mu_1\mu_2}(\tau), ..., X^{\mu_1\mu_2\mu_3\mu_4}(\tau)$ . They are a generalization of the center of mass coordinates in the sense that they provide information about the object 2-vector, 3-vector, and 4-vector extension and orientation<sup>1</sup>.

Instead of one parameter  $\tau$  we can introduce two parameters  $\tau$  and  $\sigma$ . Usual strings are described by the mapping  $(\tau, \sigma) \to x^{\mu} = X^{\mu}(\tau, \sigma)$ , where the embedding functions  $X^{\mu}(\tau, \sigma)$  describe a 2-dimensional worldsheet swept by a string. This can be generalized to C-space. So we obtain generalized strings, considered in refs. [16, 17], described by polyvector variables  $X^{M}(\tau, \sigma)\gamma_{M}$ . According to the procedure described above, the latter polyvector can be expanded as as sum of bosonic or fermionic fields. Altogether there are 16 real degrees of freedom incorporated in bosonic fields  $X^{M}(\tau, \sigma)$ , or equivalently in fermionic fields  $\theta^{\tilde{A}}(\tau, \sigma)$ . According to this approach we do not need a higher dimensional target spacetime for a consistent formulation of (quantized) string theory. Instead of a higher dimensional space we have Clifford space.

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<sup>&</sup>lt;sup>1</sup>A systematic and detailed treatment is in ref. [6].

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